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MARTINGALE TRANSFORMS AND RELATED SINGULAR INTEGRALS

BY

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ABSTRACT. The operators obtained by taking conditional expectation of continuous time martingale transforms are studied, both on the circle T and on $\mathbf{R}^n$. Using a Burkholder-Gundy inequality for vector-valued martingales, it is shown that the vector formed by any number of these operators is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, with constants that depend only on p and the norms of the matrices involved. As a corollary we obtain a recent result of Stein on the boundedness of the Riesz transforms on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, with constants independent of n.

0. Introduction. For $f \in L^p(\mathbf{R}^n)$, $1 \leqslant p < \infty$, we define the Riesz transforms by

$$R_j f(x) = \lim_{\varepsilon \to 0} \pi^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \int_{|y|>\varepsilon} \frac{y_j f(x-y)}{|y|^{n+1}}\, dy$$

for $j = 1, 2, \ldots, n$. These operators are the basic singular integrals in $\mathbf{R}^n$ and it is well known (see **[14]**) that if we set

$$Rf(x) = \left(\sum_{j=1}^{n} |R_j f(x)|^2 \right)^{1/2},$$

then this operator has the strong type inequality

$$\|Rf\|_{L^p(\mathbf{R}^n)} \leqslant C_{p,n} \|f\|_{L^p(\mathbf{R}^n)}, \qquad 1 < p < \infty, \tag{1}$$

and the weak type inequality

$$m\{x \colon Rf(x) > \lambda\} \leqslant \frac{C_n}{\lambda} \|f\|_{L^1(\mathbf{R}^n)}, \tag{2}$$

where the constants $C_{p,n}$ and C_n depend on the parameters indicated.

There has been substantial interest recently in studying the behavior of such constants in classical operators in analysis as $n \to \infty$. In particular, Stein and Strömberg **[16]** have shown that for the basic Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy$$

(here $B(x, r) = \{y \colon |x - y| < r\}$), we have

$$\|Mf\|_{L^p(\mathbf{R}^n)} \leqslant C_p \|f\|_{L^p(\mathbf{R}^n)}, \qquad 1 < p < \infty, \tag{3}$$

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and

$$m\{x: Mf(x) > \lambda\} \leqslant \frac{Cn}{\lambda}\|f\|_{L^1(\mathbf{R}^n)} \tag{4}$$

with C_p and C independent of n.

Using (3) Stein [**15**] has shown that the constant $C_{p,n}$ in the strong type inequality (1) can also be taken to be independent of n. In his closing remarks, Stein suggests that notions from probability theory may be helpful in the further understanding of analysis results as $n \to \infty$. In this paper we show this is indeed the case. Using a probabilistic interpretation of the Riesz transforms given by Gundy and Varopoulos [**10**], and techniques from the theory of martingales, we give a simple proof of Stein's result. The three key points in our proof are: (1) the classical inequalities for Brownian martingales do not depend on the dimension of the Brownian motion; (2) the Riesz transforms are conditional expectations of martingale transforms with matrices which have norms that do not grow with the dimension; and (3) conditional expectation is a contraction in L^p.

We have organized this paper as follows. In §1, we define martingale transforms on the Brownian filtration, state their basic properties and prove a Burkholder-Gundy inequality for vector-valued martingales. In §2, we begin to connect martingale transforms to analytical objects and define a collection of operators on the circle which generalize the conjugation operator. These operators are obtained by taking conditional expectation of martingale transforms. Several propositions are proved which describe their basic properties. In particular, it is shown that our operators are singular integrals and we give an explicit formula for their kernels. All of the results in this section remain valid if we replace the unit circle in the complex plane with the unit sphere in $\mathbf{R}^n$. In §3, we study these operators in $\mathbf{R}^n$ and prove Theorem 3.1. This theorem gives a vector-valued inequality similar to (2) for our operators with constants depending only on p and the norms of the matrices defining the operators. Stein's result is obtained as a corollary of Theorem 3.1. For a single Riesz transform we show that the constant we obtain is $\sim 2\sqrt{2}p$ as $p \to \infty$.

1. Definitions and preliminary results. Let B_t be an n-dimensional Brownian motion. It is well known (see [7, §2.14]) that if X is a random variable in $L^2(\mathscr{F}_\infty)$, $\mathscr{F}_\infty = \sigma(B_t;\ t \geqslant 0)$, then X can be written as

$$X = E(X) + \int_0^\infty H_s \cdot dB_s, \tag{1.1}$$

where H_s is a process with values in $\mathbf{R}^n$ which is adapted to the Brownian filtration. That is, H_s is measurable with respect to $\mathscr{F}_s = \sigma(B_t;\ t \leqslant s)$ and has

$$E\int_0^\infty |H_s|^2\, ds < \infty. \tag{1.2}$$

Given the representation in (1.1) we can define for any real $n \times n$ matrix A the martingale transform $A * X$ of X by

$$A * X = \int_0^\infty AH_s \cdot dB_s$$

and it follows from the isometry property of the stochastic integral that

$$E|A*X|^2 = E\int_0^\infty |AH_s|^2\,ds \leqslant \|A\|^2 E\int_0^\infty |H_s|^2\,ds < \infty,$$

where $\|A\| = \sup\{|Ax|: |x| \leqslant 1\}$. Thus the martingale transform is a new random variable in $L^2(\mathscr{F}_\infty)$.

For $p \neq 2$, the isometry of the stochastic integral is replaced by the Burkholder-Gundy inequalities. First we give a definition. For X as in (1.1) we define

$$\langle X\rangle = \int_0^\infty |H_s|^2\,ds$$

and call this new random variable the area function or variance process of X. If we denote by X_t and $\langle X\rangle_t$ the stochastic integrals above up to time t, then X_t is a martingale and $\langle X\rangle_t$ is the unique increasing process which makes $X_t^2 - \langle X\rangle_t$ a martingale. We now have

(1.3) Suppose $EX = 0$ and $1 < p < \infty$. There exists constants a_p and A_p which depend only on p such that

$$a_p\left(E\langle X\rangle^{P/2}\right)^{1/p} \leqslant \left(E|X|^P\right)^{1/P} \leqslant A_p\left(E\langle X\rangle^{P/2}\right)^{1/P}.$$

When $p = 1$ we have

(1.4) Suppose X and Y are two random variables such that $\langle X\rangle_t \leqslant \langle Y\rangle_t$ for all $t > 0$. Then

$$P\left\{\sup_{t>0}|X_t| > \lambda\right\} \leqslant \frac{2}{\lambda}E|Y|.$$

For the proof of (1.4) see Burkholder [3] where it is shown that 2 is the best constant in this inequality. The inequalities in (1.3) are by now classical and several proofs exist. Davis [5] gave a remarkable proof which identifies the best possible values for a_p and A_p. Let $D_p(x)$ be the parabolic cylinder function of parameter p, and let $M_p(Z) = M(-p/2, 1/2, Z^2/2)$ be the confluent hypergeometric function. (See Abramowitz and Stegun [1] as a general reference for these functions.) Let Z_p^* be the smallest positive zero of M_p and let Z_p be the largest zero of D_p. Davis showed that the best value for a_p is Z_p^* for $p \geqslant 2$ and Z_p for $1 < p < 2$. The best value for A_p is Z_p for $p \geqslant 2$ and Z_p^* for $1 < p \leqslant 2$. We shall need to use this fact later when we estimate the constant for the Riesz transforms.

We now observe that $\langle A*X\rangle_t \leqslant \|A\|^2\langle X\rangle_t$ and therefore it follows from (1.3) and (1.4) that

$$\|A*X\|_p \leqslant \|A\|\frac{A_p}{a_p}\|X\|_p, \qquad 1 < p < \infty, \tag{1.5}$$

and

$$P\{|A*X| > \lambda\} \leqslant 2\frac{\|A\|}{\lambda}E|X|. \tag{1.6}$$

For our applications we need to prove a generalization of (1.3). First recall the following

LEMMA (GARSIA [8]). *Let A_t be a positive continuous increasing process with $A_0 = 0$. If there is a positive random variable Y such that $E(A_\infty - A_T|\mathscr{F}_T) \leqslant E(Y|\mathscr{F}_T)$ for any stopping time T, then $E(A_\infty^p) \leqslant p^pE(Y^p)$, $1 < p < \infty$.*

For the proof of this Lemma see Lenglart, Lepingle and Pratelli [**11**].

THEOREM 1.1. *Let* $X^i = \int_0^\infty H_s^i \cdot dB_s$, $1 \leqslant i \leqslant m$, *be any sequence of random variables in* $L^p(\mathscr{F}_\infty)$, *with* $p \geqslant 2$. *Then there are constants* C_p' *and* C_p'' *which depend only on p so that*

$$C_p' E\left(\sum_{i=1}^m |X^i|^2\right)^{p/2} \leqslant E\left(\sum_{i=1}^m \langle X^i \rangle\right)^{p/2} \leqslant C_p'' E\left(\sum_{i=1}^m |X^i|^2\right)^{p/2}.$$

In fact, $C_p'' = (p/2)^{p/2}$.

PROOF. Recall that $\langle X^i \rangle_t = \int_0^t |H_s^i|^2\, ds$ is the positive continuous adapted increasing process which makes $(X_t^i)^2 - \langle X^i \rangle_t$ a martingale. Thus for any stopping time T we have

$$E\big(\langle X^i \rangle - \langle X^i \rangle_T | \mathscr{F}_T\big) = E\Big((X^i)^2 - (X_T^i)^2 | \mathscr{F}_T\Big) \leqslant E\Big((X^i)^2 | \mathscr{F}_T\Big).$$

Summing both sides of the previous inequality we see that

$$E\left(\sum_{i=1}^m \langle X^i \rangle - \sum_{i=1}^m \langle X^i \rangle_T | \mathscr{F}_T\right) \leqslant E\left(\sum_{i=1}^m |X^i|^2 | \mathscr{F}_T\right)$$

and applying Garsia's Lemma with $p/2 > 1$ we have

$$E\left(\sum_{i=1}^m \langle X^i \rangle\right)^{p/2} \leqslant (p/2)^{p/2} E\left(\sum_{i=1}^m |X^i|^2\right)^{p/2}. \tag{1.7}$$

Note that for $p = 2$, (1.7) is trivial and therefore we have the right-hand side inequality in our theorem for $p \geqslant 2$ as we wanted.

To prove the left-hand side inequality let $\varepsilon > 0$ be given. For $p \geqslant 1$ consider the function F: $\mathbf{R}^m \to \mathbf{R}$ defined by $F(x) = (\varepsilon + \sum_{i=1}^m |x_i|^2)^{p/2} = |\bar{x}|^p$. Since $\varepsilon > 0$ the function F is C^∞ and we can apply Itô's formula to conclude

$$F(M_t) - F(M_0) = \sum_{i=1}^m \int_0^t D_i F(M_s)\, dX_s^i + \frac{1}{2} \sum_{i,j} \int_0^t D_{ij} F(M_s)\, d\langle X^i, X^j \rangle_s,$$

where $M_t = (\varepsilon + \sum_{i=1}^m |X_t^j|^2)^{1/2}$. Also $D_i F(x) = p x_i |\bar{x}|^p$ and

$$D_{ij} F(x) = p|x|^{p-2} \delta_{ij} + p(p-2) x_i x_j |\bar{x}|^{p-4}, \tag{1.8}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. By the Cauchy-Schwarz inequality,

$$\sum_{i,j} X_s^i X_s^j H_x^i \cdot H_s^j \leqslant |M_s|^2 \sum_{i=1}^m |H_s^i|^2. \tag{1.9}$$

So if $p = 1$, (1.8) and (1.9) imply

$$\frac{1}{2} \sum_{ij} \int_0^t D_{ij} F(M_s)\, d\langle X^i, X^j \rangle_s \geqslant 0.$$

Therefore M_t is a submartingale.

As before, the inequality we want to prove is trivial for $p = 2$. So assume $p > 2$. By (1.8) and (1.9)

$$\begin{aligned}\frac{1}{2}\int_0^\infty \sum_{ij} D_{ij}F(M_s)\, d\langle X^i, X^j\rangle_s &\leqslant \frac{1}{2}p\int_0^\infty |M_s|^{p-2}\sum_{i=1}^m d\langle X^i\rangle_s \\ &\quad + \frac{1}{2}p(p-2)\int_0^\infty |M_s|^{p-2}\sum_{i=1}^m d\langle X^i\rangle_s \\ &= \frac{1}{2}p(p-1)\int_0^\infty |M_s|^{p-2}\sum_{i=1}^m d\langle X^i\rangle_s.\end{aligned}$$

Taking expectation of the previous inequality and setting $M^* = \sup_t|M_t|$,

$$\begin{aligned}E|M|^p - \varepsilon^{p/2} &\leqslant \frac{1}{2}p(p-1)E\left(\int_0^\infty |M_s|^{p-2}\sum_{i=1}^m d\langle X^i\rangle_s\right) \\ &\leqslant \frac{1}{2}p(p-1)E\left(|M^*|^{p-2}\sum_{i=1}^m \int_0^\infty d\langle X^1\rangle_s\right) \\ &= \frac{1}{2}p(p-1)E\left(|M^*|^{p-2}\sum_{i=1}^m \langle X^i\rangle\right).\end{aligned}$$

Applying Hölder's inequality with exponents $p/2$ and $p/(p-2)$ we have

$$\begin{aligned}E|M|^p - \varepsilon^{p/2} &\leqslant \frac{1}{2}p(p-1)\left[E|M^*|^p\right]^{(p-2)/p}\left[E\left(\sum_{i=1}^m \langle X^i\rangle\right)^{p/2}\right]^{2/p} \\ &\leqslant C_p\left[E|M|^p\right]^{(p-2)/p}\left[E\left(\sum_{i=1}^m \langle X^i\rangle\right)^{p/2}\right]^{2/p},\end{aligned}$$

where the last inequality follows from Doob's maximal inequality applied to the submartingale M_t. The constant C_p depends only on p.

If we now let $\varepsilon \to 0$ we get

$$E\left(\sum_{i=1}^m |X^i|^2\right)^{p/2} \leqslant C_p\left[E\left(\sum_{i=1}^m |X^i|^2\right)^{p/2}\right]^{(p-2)/p}\left[E\left(\sum_{i=1}^m \langle X^i\rangle\right)^{p/2}\right]^{2/p}$$

from which the left-hand side of the theorem follows.

2. Projection of martingale transforms on T. Let $D = \{Z \in \mathbf{C}\colon |Z| < 1\}$ be the unit disc in the complex plane and let $T = \partial D$ be the unit circle equipped with the probability measure $dm = d\theta/2\pi$. Let B_t be a two-dimensional Brownian motion starting at the origin and let $\tau = \inf\{t\colon |B_t| = 1\}$. For $f \in L^2(T)$, we let $u(Z) = E_Z(f(B_\tau))$ be its harmonic extension to D. Itô's formula implies

$$f(B_\tau) = u(0) + \int_0^\tau \nabla u(B_s)\cdot dB_s. \tag{2.1}$$

Changing notation, $X = f(B_\tau)$ and $H_s = \nabla u(B_s)1_{(s<\tau)}$, converts (2.1) into

$$X = EX + \int_0^\infty H_s\cdot dB_s \tag{2.2}$$

and if A is any 2×2 matrix we define the martingale transform of $f(B_\tau)$ as in the previous section.

If we take $H = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ it follows from the Cauchy-Riemann equations and the Itô formula that $H * f(B_\tau) = \tilde{f}(B_\tau)$, where $\tilde{f}$ is the conjugate function of f. That is, if $u(Z) = E_Z(f(B_\tau))$ and $\tilde{u}(Z) = E_Z(\tilde{f}(B_\tau))$, then $u + i\tilde{u}$ is analytic in D and $\tilde{u}(0) = 0$. Since $\|H\| = 1$, it follows from (1.5) and (1.6) that $E|\tilde{f}(B_\tau)|^p \leqslant (A_p/a_p)^p E|f(B_\tau)|$, $1 < p < \infty$, and $P\{|\tilde{f}(B_\tau)| > \lambda\} \leqslant (2/\lambda)E|f(B_\tau)|$. Since B_τ has the uniform distribution on T, these inequalities imply the classical Theorems of M. Riesz and Kolmogorov: $\|\tilde{f}\|_p \leqslant (A_p/a_p)\|f\|_p$, $1 < p < \infty$, and $m\{|\tilde{f}| > \lambda\} \leqslant (2/\lambda)\|f\|_1$.

The argument just presented works equally well in more general domains which are not necessarily simply connected. More precisely, let G be a bounded domain in the complex plane and fix $Z_0 \in G$. Let B_t be a two-dimensional Brownian motion starting at Z_0 and $\tau = \inf\{t > 0\colon B_t \notin G\}$. Suppose $u + i\tilde{u}$ is analytic in G and continuous in $G \cup \partial G$ with $\tilde{u}(Z_0) = 0$. The same argument above shows that $E_{Z_0}|\tilde{u}(B_\tau)|^p \leqslant (A_p/a_p)^p E_{Z_0}|u(B_\tau)|^p$, $1 < p < \infty$, and $P_{Z_0}\{|\tilde{u}(B_\tau)| \geqslant \lambda\} \leqslant (2/\lambda)E_{Z_0}|u(B_\tau)|$. If we denote by dW_{Z_0} the harmonic measure on ∂G with respect to Z_0, we can write the previous inequalities as

$$\int_{\partial G} |\tilde{u}(\xi)|^p \, dW_{Z_0}(\xi) \leqslant (A_p/a_p)^p \int_{\partial G} |u(\xi)|^p \, dW_{Z_0}(\xi), \qquad 1 < p < \infty,$$

and

$$W_{Z_0}\{\xi \in \partial G\colon |\tilde{u}(\xi)| > \lambda\} \leqslant \frac{2}{\lambda} \int_{\partial G} |u(\xi)| \, dW_{Z_0}(\xi).$$

Let us again restrict our attention to the unit disc. We begin by observing that if A is an arbitrary 2×2 matrix, then $A\nabla u$ will not be the gradient of a harmonic function and so $A * f(B_\tau)$ will not be a function of B_τ. To turn the random variable $A * f(B_\tau)$ into a function defined on T we take conditional expectation and define the operator T_A by

$$(T_A f)(B_\tau) = E\left(\int_0^\tau A\nabla u(B_s) \cdot dB_s | B_\tau\right)$$

or less formally

$$(T_A f)(e^{i\theta}) = E\left(\int_0^\tau A\nabla u(B_s) \cdot dB_s | B_\tau = e^{i\theta}\right).$$

The T_A's give a family of operators on the circle which generalize the conjugation operator discussed above. In the next few propositions we prove some of the basic properties of these operators. We begin with a simple observation: T_A is a bounded operator on $L^p(T)$ for $1 < p < \infty$. To see this observe that from the boundedness of the martingale transforms, (1.5), we have

$$\begin{aligned} E\left|\int_0^\tau A\nabla u(B_s) \cdot dB_s\right|^p &\leqslant \|A\|^p \left(\frac{A_p}{a_p}\right)^p E|f(B_\tau)|^p \\ &= \|A\|^p \left(\frac{A_p}{a_p}\right)^p \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta, \end{aligned}$$

where the last inequality follows since B_τ has the uniform distribution on T. Since conditional expectation is a contraction on L^p, T_A is bounded and in fact, the operator norm $\|T_A\| \leqslant \|A\| A_p/a_p$. So that when $p = 2$, the norm of T_A is less than the norm of A. Another observation, which the interested reader can verify, is that the adjoint of the operator T_A, as an operator in $L^2(T)$, is given by T_{A^*} where A^* is the transpose of A.

PROPOSITION 2.1. *The operator T_A is a singular integral on the circle whose kernel can be computed explicitly.*

PROOF. Let $h_\theta(Z) = (1 - |Z|^2)/|e^{i\theta} - Z|^2$, $Z \in D$, denote the Poisson kernel, or in probabilistic terms, the probability density (module dividing by 2π) of exiting D at $e^{i\theta}$ starting at Z. For $0 < r < 1$ we define the stopping time $\tau_r = \inf\{t > 0: |B_t| = r\}$. τ_r increases to τ as r increases to 1 and for $f \in L^2(T)$ we can write

$$T_A f(e^{i\theta}) = \lim_{r \uparrow 1} E\left(\int_0^{\tau_r} A\nabla u(B_s) \cdot dB_s | B_\tau = e^{i\theta}\right). \tag{2.3}$$

In this way we can write $T_A f$ as a limit of h-transforms (see Durrett [7, Chapter 3]). In other words, for every $0 < r < 1$,

$$E\left(\int_0^{\tau_r} A\nabla u(B_s) \cdot dB_s | B_\tau = e^{i\theta}\right) = \frac{1}{h(0)} E\left(\int_0^{\tau_r} A\nabla u(B_s) \cdot dB_s h_\theta(B_{\tau_r})\right).$$

Since $h_\theta(Z)$ is also harmonic we have by Itô's formula

$$h_\theta(B_{\tau_r}) = 1 + \int_0^{\tau_r} \nabla h_\theta(B_s) \cdot dB_s.$$

This and the formula for the covariance of stochastic integrals combined with the occupation time density formula (see [7, Chapter 1]) give

$$\begin{aligned} E\left(\int_0^{\tau_r} A\nabla u(B_s) \cdot dB_s | B_\tau = e^{i\theta}\right) &= E\left(\int_0^{\tau_r} A\nabla u(B_s) \cdot \nabla h_\theta(B_s)\, ds\right) \\ &= \frac{1}{\pi}\int_{D_r} \log\left(\frac{r}{Z}\right) A\nabla u(Z) \cdot \nabla h_\theta(Z)\, dZ, \end{aligned} \tag{2.4}$$

where $D_r = \{Z \in \mathbf{C}: |Z| < r\}$. However, since

$$u(Z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi}) h_\phi(Z)\, d\phi,$$

we have for $|Z| < r < 1$,

$$A\nabla u(Z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi}) A\nabla h_\phi(Z)\, d\phi.$$

Substituting this in (2.4) gives

$$\begin{aligned} &E\left(\int_0^{\tau} \nabla u(B_s) \cdot dB_s | B_\tau = e^{i\theta}\right) \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{\pi}\int_{D_r} \log\frac{r}{|Z|} A\nabla h_\phi(Z) \cdot \nabla h_\theta(Z)\, dZ\right) f(e^{i\phi})\, d\phi. \end{aligned}$$

Setting

$$k_A^r(e^{i\phi}, e^{i\theta}) = \frac{1}{\pi}\int_{D_r} \log\frac{r}{|Z|} A\nabla h_\phi(Z)\cdot\nabla h_\theta(Z)\,dZ$$

we can write (2.3) as

$$T_A f(e^{i\theta}) = \lim_{r\uparrow 1} \frac{1}{\pi}\int_0^{2\pi} k_A^r(e^{i\phi}, e^{i\theta}) f(e^{i\phi})\,d\phi$$

which is the conclusion of our proposition.

To check our formula the reader should observe that

$$\begin{aligned} k_A^r(e^{i\phi}, e^{i\theta}) &= \frac{1}{\pi}\int_{D_r} \log\frac{r}{|Z|} A\nabla h_\phi(Z)\cdot\nabla h_\theta(Z)\,dZ \\ &= E\left(\int_0^{\tau_r} A\nabla h_\phi(B_s)\cdot\nabla h_\theta(B_s)\,ds\right) \\ &= E\left(\int_0^{\tau_r} A\nabla h_\phi(B_s)\cdot dB_s | B_\tau = e^{i\theta}\right). \end{aligned}$$

So that if $A = H$

$$\begin{aligned} k_H^r(e^{i\phi}, e^{i\theta}) &= E\left(\int_0^{\tau_r} \tilde{h}_\phi(B_s)\cdot dB_s | B_\tau = e^{i\theta}\right) \\ &= \tilde{h}_\phi(re^{i\theta}) = \frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2} \end{aligned}$$

and

$$\lim_{r\uparrow 1} k_H^r(e^{i\phi}, e^{i\theta}) = \cot\left(\frac{\theta-\phi}{2}\right)$$

which is the kernel for the conjugate operator.

Let us denote by $M(2, R)$ the set of all 2×2 matrices with real coefficients and by $\mathscr{L}(L^2(T))$ the set of all continuous linear operators on $L^2(T)$. We have

PROPOSITION 2.2. *The linear map from $M(2,\mathbf{R})$ to $\mathscr{L}(L^2(T))$ given by $A\to T_A$ is one-to-one.*

PROOF. Let $f\in L^2(T)$ and as usual u is its harmonic extension to D. Suppose $T_A f\equiv 0$. Then

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} T_A f(e^{i\theta}) f(e^{i\theta})\,d\theta \\ &= E(T_A f(B_\tau) f(B_\tau)) = E\left[E\left(\int_0^\tau A\nabla u(B_s)\cdot dB_s | B_\tau\right) f(B_\tau)\right] \\ &= E\left(\int_0^\tau A\nabla u(B_s)\cdot\nabla u(B_s)\,ds\right) \\ &= \frac{1}{\pi}\int_D \log\frac{1}{|Z|} A\nabla u(Z)\cdot\nabla u(Z)\,dz, \end{aligned}$$

where we have used again the formula for the covariance of stochastic integrals and the occupation time density formula. If we now take $f = \cos\theta$ we have $u = x$ and $\nabla u = (1, 0)$. If $A = (a_{ij})$, then $A\nabla u = (a_{11}, a_{21})$ and $A\nabla u \cdot \nabla u = a_{11}$. Thus

$$a_{11}\int_D \log\frac{1}{|Z|}\,dZ = 0 \Rightarrow a_{11} = 0.$$

If we take $f = \sin\theta$ we have $u = y$, $\nabla u = (0,1)$, $A\nabla u \cdot \nabla u = a_{22}$ and as before this shows $a_{22} = 0$. Testing with $f = \sin\theta \pm \cos\theta$ we get $a_{12} + a_{21} = 0$ and $a_{12} - a_{21} = 0$ from which we conclude $a_{12} = a_{21} = 0$. So A is the zero matrix and this proves the proposition.

After the representation given by Proposition 2.1 it is natural to ask if our operators are convolution operators. The next proposition gives a very precise answer to this question.

PROPOSITION 2.3. *In general $T_A T_B \neq T_{AB}$ but if H is the matrix given the conjugate operator, then $T_H T_A = T_{HA}$ and $T_A T_H = T_{AH}$. Also, T_A is a Fourier multiplier in $L^2(T)$ if and only if $A = \alpha I + \beta H$, α and β real numbers.*

PROOF. Since $T_H f = \tilde{f}$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} T_A T_H f(e^{i\theta})g(e^{i\theta})\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} T_A\tilde{f}(e^{i\theta})g(e^{i\theta})\,d\theta \\ &= E(T_A\tilde{f}(B_\tau)g(B_\tau)) = E\left(E\left(\int_0^\tau A\nabla\tilde{u}(B_s)\cdot dB_s|B_\tau\right)g(B_\tau)\right) \\ &= E\left(E\left(\int_0^\tau AH\nabla u(B_s)\cdot dB_s|B_\tau\right)g(B_\tau)\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} T_{AH}f(e^{i\theta})g(e^{i\theta})\,d\theta.\end{aligned}$$

Since this holds for any $g \in L^2(T)$ it follows that $T_A T_H = T_{AH}$.

Next recall that the adjoint of the conjugate operator is $(-)$ itself and $H^* = -H$. We have as above

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} T_{HA}f(e^{i\theta})g(e^{i\theta})\,d\theta &= E\left(\int_0^\tau HA\nabla u_f(B_s)\cdot\nabla u_g(B_s)\,ds\right) \\ &= -E\left(\int_0^\tau A\nabla u_f(B_s)\cdot H\nabla u_g(B_s)\,ds\right) \\ &= -E\left(\int_0^\tau A\nabla u_f(B_s)\cdot\nabla\tilde{u}_g(B_s)\,ds\right) = -\frac{1}{2\pi}\int_0^{2\pi} T_A f(e^{i\theta})\tilde{g}(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} T_H T_A f(e^{i\theta})g(e^{i\theta})\,d\theta\end{aligned}$$

and therefore $T_{HA} = T_H T_A$.

For the second part of our proposition let us recall that an operator S: $L^2(T) \to L^2(T)$ is called a Fourier multiplier with symbol $\sigma(n) \in l^\infty$ if $(Sf)^\wedge(n) = \sigma(n)\hat{f}(n)$ for all $f \in L^2(T)$. Here $\hat{f}$ means the Fourier transform of f. A well-known example of such an operator is the conjugation operator $T_H f = \tilde{f}$ whose symbol is given by

$\sigma_H(n) = -i\,\mathrm{sign}(n)$. So if σ_A is the symbol for T_A we must have

$$(T_A T_H f)^\wedge(n) = \sigma_A(n)\sigma_H(n)\hat{f}(n) = \sigma_H(n)\sigma_A(n)\hat{f}(n) = (T_H T_A f)^\wedge(n).$$

This shows that $T_A T_H = T_{HA}$. From the first part of the proposition we have $T_{HA} = T_{AH}$ or $T_{HA-AH} = 0$. From Proposition 2.2 we may conclude that this happens if and only if $HA = AH$ and it follows that $A = \alpha I + \beta H$. So T_A is a multiplier operator (hence a convolution operator) if and only if $A = \alpha I + \beta H$, which gives the proposition.

We will now show that if we enlarge our collection of matrices by allowing variable coefficients it is possible to obtain convolution operators from our operators T_A. Let $f \in L^2(T)$ and denote by f_θ the function f rotated by the angle θ. That is, $f_\theta(e^{i\phi}) = f(e^{i(\theta+\phi)})$. Let u_θ be the corresponding harmonic extension. If we let R_θ be the 2×2 matrix which represents rotation by θ we have, by a trivial change of variables and the rotation invariance property of Brownian motion, that

$$\begin{aligned} T_A f(e^{i\theta}) &= E\left(\int_0^\tau R_{-\theta} A R_\theta \cdot \nabla u_\theta(B_s) \cdot dB_s \middle| B_\tau = 1\right) \\ &= \left(T_{R_{-\theta} A R_\theta f_\theta}\right)(1). \end{aligned}$$

Since $R_{-\theta} R_\theta = I$, we see that if A commutes with rotations, then our operators T_A also commute with rotations. We know from above that this happens only when A is a linear combination of I and H. However, if we allow A to have variable coefficients the situation changes. For example, let $A(Z) = \varphi(Z)H$, where $\varphi(Z)$ is a real valued function defined on D which is rotationally invariant and has $\sup_{Z \in D}|\varphi(Z)| \leqslant M$. Then the operator

$$(T_{A(Z)} f)(e^{i\theta}) = E\left(\int_0^\tau \varphi(B_s) H \nabla u(B_s) \cdot dB_s \middle| B_\tau = e^{i\theta}\right)$$

still defines a continuous linear operator on $L^2(T)$ (any $1 < p < \infty$) and since $R_{-\theta} A R_\theta = A(Z)$ for all $Z \in D$, we have $(T_{A(Z)} f)(e^{i\theta}) = (T_{A(Z)} f_\theta)(1)$, and in general $(T_{A(Z)} f)_\theta(e^{i\phi}) = (T_{A(Z)} f_\theta)(e^{i\phi})$. If we now recall that every continuous operator from $L^2(T)$ into itself which commutes with rotations is a multiplier operator (see Stein [**14**, p. 28]), we have that $T_{A(Z)}$ is a convolution operator on $L^2(T)$. The kernel can be computed explicitly as we did above for the case of constant coefficients.

REMARK 2.1. As mentioned in the introduction most of the results proved in this section remain valid if we replace the unit disc in the complex plane by the unit ball in $\mathbf{R}^n$. Of particular interest is the case of even dimensions because of its possible connections to several complex variables. We illustrate this with one example. Let $B = \{Z \in \mathbf{C}^n\colon |Z| < 1\}$ be the unit ball in $\mathbf{C}^n$, $S = \partial B$ the unit sphere and σ the normalized surface measure on S. Suppose $f = u + i\tilde{u}$ is in $H^p(S)$, $1 < p < \infty$, and $\tilde{u}(0) = 0$ (see Rudin [**13**, p. 87] for the definition of $H^p(S)$). Then,

$$\int_S |\tilde{u}(\xi)|^p \, d\sigma(\xi) \leqslant \left(\frac{A_p}{a_p}\right)^p \int_S |u(\xi)|^p \, d\sigma(\xi). \tag{2.5}$$

The constants A_p and a_p are the same as in (1.3) and in particular they are independent of the dimension. To prove (2.5) we simply observe that if H is the $2n \times 2n$ matrix defined by

$$H = \left[\begin{array}{cc|c|ccc|cc} 0 & -1 & & & & & & \\ 1 & 0 & 0 & & & & & \\ \hline & 0 & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & 0 & & \\ \hline & & & & 0 & 0 & -1 & \\ & & & & & 1 & 0 & \end{array}\right],$$

then the Cauchy-Riemann equations and Itô's formula imply that $H * u(B_\tau) = \tilde{u}(B_\tau)$. (2.5) is therefore a special case of (1.5) and we are done. (For the classical proof of this result see [**13**, p. 125].)

REMARK 2.2. It is not difficult to show that our operators, like the classical Calderón-Zygmund singular integrals, are also bounded on BMO(T) and hence on $H^1(T)$. An interesting question is whether these operators can be used to characterize $H^1(T)$. Most likely this is the case provided that the matrices used do not have a common real eigenvector. Our guess is based on the fact that martingale transforms characterize H^1-martingales if and only if the matrices do not have a common real eigenvector. This theorem is due to S. Janson and the reader can find a proof in [**7**, p. 167].

3. Projection of martingale transforms on $\mathbf{R}^n$. Fix $y > 0$ and let B_t be an $(n+1)$-dimensional Brownian motion with initial distribution $m_y = m \otimes \delta_y$, the Lebesgue measure on the hyperplane $\{(x, y)\colon x \in \mathbf{R}^n\}$. In other words, B_t is a process whose distribution is given by

$$P^y(A) = \int_{\mathbf{R}^n} P_{(x,y)}(A)\,dx,$$

where $P_{(x,y)}$ is the distributed of Brownian motion starting at $(x, y) \in \mathbf{R}^{n+1}$. Let $\mathbf{R}^{n+1}_+ = \{(x, y')\colon x \in \mathbf{R}^n,\ y' > 0\}$ be the upper half space and as usual we view $\mathbf{R}^n$ as its boundary. If $\tau = \inf\{t > 0\colon B_t \notin \mathbf{R}^{n+1}_+\}$ and f is a positive Borel function in $\mathbf{R}^n$, we have

$$E^y(f(B_\tau)) = \int_{\mathbf{R}^n} E_{(x,y)}(f(B_\tau))\,dx = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} k_y(\theta, x) f(\theta)\,d\theta\,dx, \tag{3.1}$$

where

$$k_y(\theta, x) = \pi^{-(n+1)/2}\,\Gamma\!\left(\frac{n+1}{2}\right)\frac{y}{\left(|x-\theta|^2 + y^2\right)^{\frac{n+1}{2}}},$$

the Poisson kernel (or Cauchy density). Therefore if we interchange the order of integration (3.1) becomes

$$E^y(f(B_\tau)) = \int_{\mathbf{R}^n} f(\theta)\,d\theta. \tag{3.2}$$

Let us denote by $\mathscr{S}(\mathbf{R}^n)$ the class of rapidly decreasing functions on $\mathbf{R}^n$. If $f \in \mathscr{S}(\mathbf{R}^n)$ we let $u(x, y) = E_{(x,y)}(f(B_\tau))$ be the Poisson integral of f. Since u is C^2 in $\mathbf{R}^{n+1}_+$ and continuous at the boundary, it follows from Itô's formula that

$$f(B_\tau) = u(B_0) + \int_0^\tau \nabla u(B_s) \cdot dB_s.$$

So for any $(n+1) \times (n+1)$ matrix we define the martingale transform of $f(B_\tau)$ as before:

$$A * f(B_\tau) = \int_0^\tau A \nabla u(B_s) \cdot dB_s.$$

It follows from (3.1) that

$$E^y|A * f(B_\tau)|^2 = \int_{\mathbf{R}^n} E_{(x,y)}|A * f(B_\tau)|^2\,dx$$

and from (1.5) that

$$E_{(x,y)}|A * f(B_\tau)|^2 \leqslant C\|A\|^2 E_{(x,y)}|f(B_\tau)|^2.$$

Integrating both sides of the previous inequality with respect to x and applying (3.2) we have

$$E^y|A * f(B_\tau)|^2 \leqslant C\|A\|^2 \|f\|_{L^2(\mathbf{R}^n)} < \infty.$$

So $A * f(B) \in L^2(P^y)$.

From the definition of P^y we see that P^y is not a probability measure. Nevertheless, if $\mathscr{F}$ is a σ-algebra and $X \in L^p(P^y)$, $1 < p < \infty$, we can still define $E^y(X|\mathscr{F})$ which is just a Radon-Nikodym derivative and it belongs to $L^p(P^y)$. Thus we can take conditional expectation of $A * f(B_\tau)$ and we define the operator T_A^y by

$$(T_A^y f)(\theta) = E^y\left(\int_0^\tau A \nabla u(B_s) \cdot dB_s \middle| B_\tau = \theta\right).$$

THEOREM 3.1. *Let $(A_1, A_2, \ldots, A_m)$ be real $(n+1) \times (n+1)$ matrices and for $f \in \mathscr{S}(\mathbf{R}^n)$ set*

$$T^y f(\theta) = \left(\sum_{i=1}^m \left|T_{A_i}^y f(\theta)\right|^2\right)^{1/2}.$$

Then

$$\|T^y f\|_{L^p(\mathbf{R}^n)} \leqslant C_p \|\vec{A}\|\, \|f\|_{L^p(\mathbf{R}^n)}, \qquad 1 < p < \infty,$$

where C_p depends only on p and

$$\|\vec{A}\|^2 = \sup\left\{\sum_{i=1}^m |A_i x|^2 : |x| \leqslant 1\right\}.$$

PROOF. Assume first $p \geqslant 2$. Since conditional expectation is a contraction on L^p, $p \geqslant 1$, we have

$$\begin{aligned}
\|T^y f(B_\tau)\|^p_{L^p(P^y)} &= E^y|T^y f(B_\tau)|^p \\
&= E^y\left(\sum_{i=1}^m \left|E^y\left(\int_0^\tau A_i\nabla u(B_s)\cdot dB_s|B_\tau\right)\right|^2\right)^{p/2} \\
&\leqslant E^y\left(\sum_{i=1}^m E^y\left(\left|\int_0^\tau A_i\nabla(B_s)\cdot dB_s\right|^2|B_\tau\right)\right)^{p/2} \\
&= E^y\left(E^y\left(\left(\sum_{i=1}^m\left|\int_0^\tau A_i\nabla u(B_s)\cdot dB_s\right|^2\right)|B_\tau\right)\right)^{p/2} \\
&\leqslant E^y\left(\sum_{i=1}^m\left|\int^\tau A_i\nabla u(B_s)\cdot dB_s\right|^2\right)^{p/2} \\
&= \int_{\mathbf{R}^n} E_{(x,y)}\left(\sum_{i=1}^m\left|\int_0^\tau A\nabla u(B_s)\cdot dB_s\right|^2\right)^{p/2} dx.
\end{aligned}$$

Applying Theorem 1.1 we get

(3.3)

$$\begin{aligned}
E_{(x,y)}\left(\sum_{i=1}^m\left|\int_0^\tau A_i\nabla u(B_s)\cdot dB_s\right|^2\right)^{p/2} &\leqslant C_p E_{(x,y)}\left(\sum_{i=1}^m\int_0^\tau|A_i\nabla u(B_s)|^2\,ds\right)^{p/2} \\
&\leqslant C_p\|\vec{A}\|^p E_{(x,y)}\left(\int_0^\tau|\nabla u(B_s)|^2\,ds\right)^{p/2}.
\end{aligned}$$

(Here and in what follows C_p is a constant depending only on p and whose value will change from line to line.) If we now apply the Burkholder-Gundy inequalities (1.3) we see that the previous expression is

$$\begin{aligned}
&\leqslant C_p\|\vec{A}\|^p E_{(x,y)}\left|\int_0^\tau \nabla u(B_s)\cdot dB_s\right|^p \\
&= C_p\|\vec{A}\|^p E_{(x,y)}|f(B_\tau) - u(x,y)|^p \\
&\leqslant C_p\|\vec{A}\|^p E_{(x,y)}|f(B_\tau)|^p,
\end{aligned}$$

where the last inequality follows since $|u(x, y)| = |E_{(x,y)}(f(B_\tau))|^p \leqslant E_{(x,y)}|f(B_\tau)|^p$. Integrating with respect to x gives

$$\|T^y f(B_\tau)\|^p_{L^p(P^y)} \leqslant C_p\|\vec{A}\|^p\int_{\mathbf{R}^n} E_{(x,y)}|f(B_\tau)|^p\,dx$$

or from (3.2)

$$\|T^y f\|_{L^p(\mathbf{R}^n)} \leqslant C_p\|\vec{A}\|\,\|f\|_{L^p(\mathbf{R}^n)}$$

and then the theorem is proved for $p \geqslant 2$.

Assume next $1 < p < 2$ and let $q > 2$ be the conjugate of p. By duality

$$\|T^y f\|_{L^p(\mathbf{R}^n)} = \sup_{\|\vec{g}\|_{L^q(\mathbf{R}^n)} \leqslant 1} \left| \int_{\mathbf{R}^n} \sum_{i=1}^{m} T_{A_i}^y f(\theta) g_i(\theta)\, d\theta \right|,$$

where $\vec{g} = (g_1, g_2, \ldots, g_m)$ and $\|\vec{g}\|_{L^q(\mathbf{R}^n)} = \|(\sum_{i=1}^m |g_i|^2)^{1/2}\|_{L^q(\mathbf{R}^n)}$. We have

$$\begin{aligned}
\left| \int_{\mathbf{R}^n} \sum_{i=1}^{m} T_{A_i}^y f(\theta) g_i(\theta)\, d\theta \right| &= \left| E^y\left(\sum_{i=1}^{m} T_{A_i}^y f(B_\tau) g_i(B_\tau) \right) \right| \\
&= \left| E^y\left(\sum_{i=1}^{m} E^y\left(\int_0^\tau A_i \nabla u(B_s) \cdot dB_s | B_\tau \right) g_i(B_\tau) \right) \right| \\
&= \left| E^y\left(\sum_{i=1}^{m} E^y\left(\int_0^\tau A_i \nabla u(B_s) \cdot dB_s g_i(B_\tau) | B_\tau \right) \right) \right| \\
&= \left| E^y \sum_{i=1}^{m} \left(\int_0^\tau A_i \nabla u(B_s) \cdot dB_s g_i(B_\tau) \right) \right| \\
&= \left| E^y \int_0^\tau \sum_{i=1}^{m} A_i \nabla u(B_s) \cdot \nabla u_i(B_s)\, ds \right|,
\end{aligned}$$

where we have used u_i to denote the Poisson integral of g_i. Using the Cauchy-Schwarz inequality the last expression is

$$\begin{aligned}
&\leqslant E^y\left(\int_0^\tau \left(\sum_{i=1}^{m} |A_i \nabla u(B_s)|^2 \right)^{1/2} \left(\sum_{i=1}^{m} |\nabla u_i(B_s)|^2 \right)^{1/2} ds \right) \\
&\leqslant E^y\left[\left(\int_0^\tau \sum_{i=1}^{m} |A_i \nabla(B_s)|^2\, ds \right)^{1/2} \left(\int_0^\tau \sum_{i=1}^{m} |\nabla u_i(B_s)|^2\, ds \right)^{1/2} \right].
\end{aligned}$$

The last inequality follows from the Kunita-Watanabe inequality. Applying Hölder's inequality the above is

$$\begin{aligned}
&\leqslant \left[E^y\left(\int_0^\tau \sum |A_i \nabla u(B_s)|^2\, ds \right)^{p/2} \right]^{1/p} \left[E^y\left(\int_0^\tau \sum_{i=1}^{m} |\nabla u_i(B_s)|^2\, ds \right)^{q/2} \right]^{1/q} \\
&\leqslant \|\vec{A}\| \left[E^y\left(\int_0^\tau |\nabla u(B_s)|^2\, ds \right)^{p/2} \right]^{1/p} \left[E^y\left(\int_0^\tau \sum_{i=1}^{m} |\nabla u_i(B_s)|^2\, ds \right)^{q/2} \right]^{1/q}.
\end{aligned}$$

The computation we did above using (1.3) shows that the first term in the previous expression is $\leqslant C_p \|\vec{A}\|\, \|f\|_{L^p(\mathbf{R}^n)}$. If we can show that the second term is $\leqslant C_p \|\vec{g}\|_{L^q(\mathbf{R}^n)}$, then we will be done. By definition

$$E^y\left(\int_0^\tau \sum_{i=1}^{m} |\nabla u_i(B_s)|^2\, ds \right)^{q/2} = \int_{\mathbf{R}^n} E_{(x,y)}\left(\int_0^\tau \sum_{i=1}^{m} |\nabla u_i(B_s)|^2\, ds \right)^{q/2} dx$$

and since $q > 2$, Theorem 1.1 shows that

$$\begin{aligned} E_{(x,y)}\left(\int_0^\tau \sum_{i=1}^m |\nabla u_i(B_s)|^2\,ds\right)^{q/2} &\leq C_q E_{(x,y)}\left(\sum_{i=1}^m \left|\int_0^\tau \nabla u_i(B_s)\cdot dB_s\right|^2\right)^{q/2} \\ &= C_q E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau) - u_i(x,y)|^2\right)^{q/2} \\ &\leq C_q E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2 + |u_i(x,y)|^2\right)^{q/2}. \end{aligned}$$

However,

$$\begin{aligned} \sum_{i=1}^m |u_i(x,y)|^2 = \sum_{i=1}^m \left|E_{(x,y)}(g_i(B_\tau))\right|^2 &\leq \sum_{i=1}^m E_{(x,y)}\left(|g_i(B_\tau)|^2\right) \\ &= E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right). \end{aligned}$$

So,

$$\begin{aligned} &E_{(x,y)}\left(\int_0^\tau \sum_{i=1}^m |\nabla u_i(B_s)|^2\,ds\right)^{q/2} \\ &\quad\leq C_q E_{(x,y)}\left[\sum_{i=1}^m |g_i(B_\tau)|^2 + \left(E_{(x,y)}\sum_{i=1}^m |g_i(B_\tau)|^2\right)\right]^{q/2} \\ &\quad\leq C_q\left[E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)^{q/2} + E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)^{q/2}\right]. \end{aligned}$$

Since $q/2 \geq 1$, Jensen's inequality gives

$$\left[E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)\right]^{q/2} \leq E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)^{q/2},$$

so

$$E_{(x,y)}\left(\int_0^\tau \sum_{i=1}^m |\nabla u_i(B_\tau)|^2\,ds\right)^{q/2} \leq C_q E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)^{q/2}$$

and integrating with respect to x gives

$$\begin{aligned} &\int_{\mathbf{R}^n} E_{(x,y)}\left(\int_0^\tau \sum_{i=1}^m |\nabla u_i(B_s)|^2\,ds\right)^{q/2} dx \\ &\quad\leq C_q\int_{\mathbf{R}^n} E_{(x,y)}\left(\sum_{i=1}^m |g_i(B_\tau)|^2\right)^{q/2} dx = C_q\|\vec{g}\|_q^q = C_p\|\vec{g}\|_q^q, \end{aligned}$$

which completes the proof of the theorem.

We will now obtain as a corollary of Theorem 3.1 the result for the Riesz transforms announced in the introduction. Let A_j, $1 \leq j \leq n$, be the $(n+1) \times (n+1)$ matrix whose entries are

$$a_{ik}^j = \begin{cases} 1 & \text{if } i = 1,\ k = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from a result of Gundy and Varopoulos [**10**] (and by different methods Gundy and Silverstein [**9**]) that $T_{A_j}^{y} f(\theta) \to \frac{1}{2} R_j f(\theta)$ a.e. as $y \to \infty$ for all $f \in \mathcal{S}(\mathbf{R}^n)$. Combining this result with Fatou's lemma, Theorem 3.1, and noticing that for this sequence of matrices $\|\vec{A}\| \leqslant 1$, we have

COROLLARY 3.2 (STEIN [**15**]). *For $f \in \mathcal{S}(\mathbf{R}^n)$,*

$$\|Rf\|_{L^p(\mathbf{R}^n)} \leqslant C_p \|f\|_{L^p(\mathbf{R}^n)}, \qquad 1 < p < \infty,$$

where $Rf(x) = (\sum_{j=1}^{m} |R_j f(x)|^2)^{1/2}$ and C_p depends only on p.

The theorem for general f in $L^p(\mathbf{R}^n)$ follows by a simple density argument.

By keeping track of the constants in the proofs of Theorems 1.1 and 3.1 we can give an explicit value for C_p but, as the interested reader can check, this constant is not very good as $p \to \infty$. If we consider only one Riesz transform the situation is much better. It follows as in the proof of Theorem 3.1 that $\|R_j f\|_{L^p(\mathbf{R}^n)} \leqslant (2A_p/a_p)\|f\|_{L^p(\mathbf{R}^n)}$, where a_p and A_p are the constants given by Davis [**5**] for the Burkholder-Gundy inequality (1.3). When $p > 2$, it follows from Abramowitz and Stegun [**1**, p. 696], that $A_p \leqslant 2(p + \frac{1}{2})^{1/2}$ and from Garsia's Lemma that $1/a_p \leqslant (p/2)^{1/2}$. So

$$\|R_j f\|_{L^p(\mathbf{R}^n)} \leqslant 2\sqrt{2p^2 + p}\, \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } p \geqslant 2.$$

The constant above is $\sim 2\sqrt{2}p$ as $p \to \infty$, which is the right order of magnitude since the best constant for the Hilbert transform $(\cot \pi/(2p))$ is $O(p)$ as $p \to \infty$ (see Pichorides [**12**]). It is also interesting to note that our constant above has the same asymptotic behavior as the best constant $(p - 1)$ for various martingale transforms given by Burkholder [**4**]. (The reader should note that the martingale transforms treated by Burkholder do not include our martingale transforms.) It would not be too surprising to the writer if the best constant for our martingale transforms turns out to be $(p - 1)$ also. This will permit us to give a better constant for the Riesz transforms than the one given here.

We end with two remarks:

(1) After this paper had been completed we learned that A. Bennett [**2**] has also given a proof of Stein's result based on the Burkholder-Gundy inequalities. His approach, however, does not use general martingale transforms and he does not give information on the behavior of the constants with respect to p. In addition, our Theorem 3.1 is more general than just the result for Riesz transforms and it works equally well if our matrices have variable entries. It would be interesting to know what operators we obtain when we project in $\mathbf{R}^n$ martingale transforms with variable coefficients.

(2) The referee has informed us that J. Duoandikoetxea and J. L. Rubio de Francia [**6**] have given yet another proof of Stein's result. Their proof does not use any square functions and it is based on the method of rotations for singular integrals with odd kernels which reduces matters to the Hilbert transform on the real line. Their constant has the same asymptotic behavior as our constant above.

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